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ON A DYNAMIC PROBLEM IN THE THEORY OF ELASTICITY

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We examine the analog of Lamb's solution for the semi-infinite plane cut. We assume that an elastic homogeneous and isotropic body, occupying the exterior of the cut $y=0, x<0$, is at rest at the beginning. At the initial instant $t=0$ an instantaneous concentrated impulse P is acting at both edges of the cut at the point of coordinate $x=-l$. This problem, with mixed boundary conditions on the semiplane, is solved by the Wiener-Hopf method. First we construct the solution for the similar stationary case, and then, on the basis of this, the solution of the nonstationary problem. By using the latter solution as Green's function, we construct the solution of the more general dynamic problem with arbitrary load distribution along the gap.

1. The stationary problem. We consider a stationary wave process in the plane xy, assuming that the physical quantities are specified in the form $f(x, y)\exp(-i\omega t)$, where $f(x, y)$ is some function, ω is the frequency of the oscillations, and t is time. The equations of the dynamic elasticity theory have the form

$$\Delta\Phi + k_1^2\Phi = 0, \quad \Delta\Psi + k_2^2\Psi = 0 \tag{1.1}$$

Here $\Phi(x, y)$ and $\Psi(x, y)$ are the potentials of the longitudinal and transversal waves, k_1 and k_2 are the wave numbers, Δ is the Laplace operator and x, y are rectangular Cartesian coordinates. The components of the stress tensor and the displacement vector are expressed in terms of the wave potentials $\Phi(x, y)$ and $\Psi(x, y)$ in the following manner:

$$\frac{\sigma_y}{2\mu} = -\left(\frac{1}{2}k_2^2 + \frac{\partial^2}{\partial x^2}\right)\Phi - \frac{\partial^2\Psi}{\partial x\partial y}$$

$$\frac{\tau_{xy}}{2\mu} = \frac{\partial^2\Phi}{\partial x\partial y} - \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_2^2\right)\Psi \tag{1.2}$$

$$v = \frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x}$$

Here σ_y and τ_{xy} are stresses, v is the component of the displacement along the y-axis, and μ is the shear modulus (the factor $\exp(-i\omega t)$ in the formulas (1.2) is omitted).

We consider the following singular boundary value problem for the semi-infinite cut:

$$\sigma_y = -P\delta(x+l)\exp(-i\omega t) \quad \text{for} \quad y=0,\ x<0$$

$$v = 0 \quad \text{for}\ y=0, \quad x>0 \tag{1.3}$$

$$\tau_{xy} = 0 \quad \text{for} \quad y=0, \quad -\infty<x<\infty$$

$$\Phi = O(r^{3/2}), \quad \Psi = O(r^{3/2}) \quad \text{for}\ r=\sqrt{x^2+y^2}\to 0$$

(the condition on the edge [1, 2]). Here $\delta(x)$ is Dirac's delta function. The problem is considered to be symmetric with respect to the x-axis.

Following [2], we seek the solution of the problem in the form

$$\Phi(x,y) = -\int_{-\infty}^{\infty} R(\lambda)\left(\frac{1}{2}k_2^2-\lambda^2\right)e^{i(\lambda x+\sqrt{k_1^2-\lambda^2}y)}\,d\lambda$$

$$\Psi(x,y) = \int_{-\infty}^{\infty} R(\lambda)\,\lambda\sqrt{k_1^2-\lambda^2}\,e^{i(\lambda x+\sqrt{k_2^2-\lambda^2}y)}\,d\lambda \qquad (1.4)$$

Here $R(\lambda)$ is the unknown function; the function $\sqrt{k^2-\lambda^2}$, analytic in the complex plane λ with two semi-infinite cuts $(-\infty, -k)$ and (k, ∞) along the real axis, is viewed as the branch of this function, real and positive for $-k<\lambda<k$, i.e. positive imaginary at the upper edge of the left cut and at the lower edge of the right cut. The integration contour for (1.4) is shown in Fig. 1.

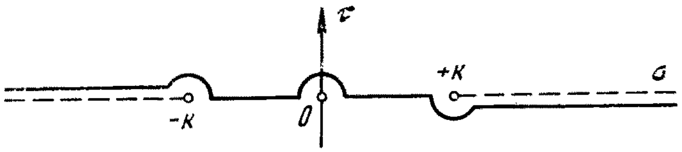

Fig. 1

The solution in the form (1.4) satisfies the wave equation (1.1) and the boundary condition $\tau_{xy}=0$ for $y=0$. The remaining boundary conditions and the condition on the edge determine the function $R(\lambda)$. Substituting (1.4) into (1.2), we obtain (for $y=0$)

$$\frac{\sigma_y}{2\mu} = \int_{-\infty}^{\infty} R(\lambda)A(\lambda)e^{i\lambda x}d\lambda, \qquad v = \int_{-\infty}^{\infty} R(\lambda)B(\lambda)e^{i\lambda x}d\lambda \qquad (1.5)$$

$$A(\lambda) = \lambda^2\sqrt{(k_1^2-\lambda^2)(k_2^2-\lambda^2)} + (\tfrac{1}{2}k_2^2-\lambda^2)^2$$

$$B(\lambda) = -\tfrac{1}{2}ik_2^2\sqrt{k_1^2-\lambda^2}$$

Applying the inverse Fourier transform to the relations (1.5), we obtain

$$A(\lambda)R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sigma_y}{2\mu}\right)_{y=0} e^{-i\lambda x}dx = \Omega^+(\lambda)+\Omega^-(\lambda)$$

$$B(\lambda)R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(v)_{y=0}e^{-i\lambda x}dx = V^+(\lambda)+V^-(\lambda) \qquad (1.6)$$

$$\Omega^+(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{0}\left(\frac{\sigma_y}{2\mu}\right)_{y=0} e^{-i\lambda x}dx, \qquad \Omega^-(\lambda) = \frac{1}{2\pi}\int_{0}^{\infty}\left(\frac{\sigma_y}{2\mu}\right)_{y=0} e^{-i\lambda x}dx$$

$$V^+(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{0}(v)_{y=0}e^{-i\lambda x}dx, \qquad V^-(\lambda) = \frac{1}{2\pi}\int_{0}^{\infty}(v)_{y=0}e^{-i\lambda x}dx$$

where the desired functions $\Omega^-(\lambda)$ and $V^+(\lambda)$ are analytic in the upper and lower half-plane of the complex variable λ, respectively. According to the boundary conditions (1.3), we have

$$\Omega^+(\lambda) = -\frac{P}{4\pi\mu} e^{i\lambda l}, \qquad V^-(\lambda) = 0 \tag{1.7}$$

Substituting (1.7) into (1.6) and eliminating $R(\lambda)$, we arrive at the following Wiener-Hopf functional equation:

$$i\left(1 - \frac{k_1^2}{k_2^2}\right)\sqrt{k_2^2 - \lambda^2}\, F(\lambda)\, V^+(\lambda) = \Omega^-(\lambda) - \frac{Pe^{i\lambda l}}{4\pi\mu}$$

$$F(\lambda) = \frac{2}{k_2^2 - k_1^2}\left[\lambda^2 + \frac{\left(\frac{1}{2}k_2^2 - \lambda^2\right)^2}{\sqrt{(k_1^2 - \lambda^2)(k_2^2 - \lambda^2)}}\right] \tag{1.8}$$

The functions $F(\lambda)$ and $\sqrt{k_2^2 - \lambda^2}$ are represented in the form

$$F(\lambda) = F^+(\lambda)\, F^-(\lambda), \qquad \sqrt{k_2^2 - \lambda^2} = \sqrt{k_2 + \lambda}\sqrt{k_2 - \lambda} \tag{1.9}$$

where the functions $F^+(\lambda)$ and $F^-(\lambda)$ are analytic and different from zero in the half-planes $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$, respectively. Due to the choice of the branch of the function $\sqrt{k_2^2 - \lambda^2}$ and of the contour which separates the lower and the upper half-planes of the λ-plane, $\sqrt{k_2 + \lambda}$ is an analytic function in the upper half-plane (cut along $y = 0,\ -k_2 > \lambda > -\infty$), while $\sqrt{k_2 - \lambda}$ in the lower half-plane (cut along $y = 0,\ \infty > \lambda > k_2$). We make use of the result concerning the factorization of the function $F(\lambda)$, given in [3]

$$F^{\pm}(\lambda) = \frac{\lambda_R \pm \lambda}{k_2 \pm \lambda}\exp\left[\frac{1}{\pi}\int_{\mp k_1}^{\mp k_2} \operatorname{arctg}\frac{\left(\frac{1}{2}k_2^2 - \xi^2\right)^2}{\xi^2\sqrt{(\xi^2 - k_1^2)(k_2^2 - \xi^2)}}\frac{d\xi}{\xi - \lambda}\right] \tag{1.10}$$

$$(\lambda_R = \omega/c_R, \quad c_R < c_2)$$

Here c_R is the propagation velocity of the Raleigh surface waves; we take, simultaneously, either the upper or the lower signs. Taking into account the factorization, the Wiener-Hopf equation (1.8) can be written in the form

$$i\left(1 - \frac{k_1^2}{k_2^2}\right)\sqrt{k_2 + \lambda}\, F^+(\lambda)\, V^+(\lambda) = \frac{\Omega^-(\lambda)}{F^-(\lambda)\sqrt{k_2 - \lambda}} - \frac{P\exp(i\lambda l)}{4\pi\mu F^-(\lambda)\sqrt{k_2 - \lambda}} \tag{1.11}$$

In order to apply the standard Wiener-Hopf technique to the Eq. (1.11), it is necessary to transform the second term in the right-hand side, since this term represents a divergent wave at infinity for $\operatorname{Im}\lambda < 0$. The function $F^-(\lambda)\sqrt{k_2 - \lambda}$ has a zero at $\lambda = \lambda_R$ and a branch point at $\lambda = k_1$. In addition, because of the behavior of the integral of Cauchy type at the extremities of the integration contour [4], the function is regular for $\lambda = k_2$. The equation (1.11), after the transformation of the second term in the right-hand side, obtains the form

$$i\left(1 - \frac{k_1^2}{k_2^2}\right)\sqrt{k_2 + \lambda}\, F^+(\lambda)\, V^+(\lambda) +$$

$$\frac{PG(\lambda)}{4\pi\mu(\lambda_R - \lambda)}\left[e^{i\lambda l}\sqrt{k_1 - \lambda} - \alpha(\lambda)e^{ik_1 l} - \beta(\lambda)e^{i\lambda_R l}\right] =$$

$$\frac{\Omega^-(\lambda)}{F^-(\lambda)\sqrt{k_2 - \lambda}} - \frac{PG(\lambda)}{4\pi\mu(\lambda_R - \lambda)}\left[\alpha(\lambda)e^{ik_1 l} + \beta(\lambda)e^{i\lambda_R l}\right]$$

$$G(\lambda) = \exp\left[-\frac{1}{\pi}\int_{k_1}^{k_2}\operatorname{arctg}\frac{\left(\frac{1}{2}k_2^2 - \xi^2\right)^2}{\xi^2\sqrt{(\xi^2 - k_1^2)(k_2^2 - \xi^2)}}\frac{d\xi}{\xi - \lambda}\right]$$

$$\alpha(\lambda) = \sqrt{\frac{k_2 - k_1}{(\lambda - k_1)(k_2 - \lambda_R)}}\, [\sqrt{k_2 - \lambda_R}\,(\sqrt{\lambda - k_1} - \sqrt{k_2 - k_1}) - \sqrt{k_2 - \lambda}\,(\sqrt{\lambda_R - k_1} - \sqrt{k_2 - k_1})] \tag{1.12}$$

$$\beta(\lambda) = \sqrt{\frac{(k_2 - \lambda)(\lambda_R - k_1)}{\lambda - k_1}}$$

The left-hand side of this equation represents a function which is analytic in the upper half-plane of the λ-plane, while the right-hand side represents a function which is analytic in the lower half-plane. According to the principle of analytic continuation we can assert that the left-hand and the right-hand sides of this equation are analytic functions, each the continuation of the other. It remains to elucidate the behavior of the so determined function, analytic in the entire plane λ, at the point at infinity. Making use of an Abelian type theorem [5] and of the condition on the edge ($\Phi = O(r^{3/2})$, $\Psi = O(r^{3/2})$ for $r = \sqrt{x^2 + y^2} \to 0$), it is easy to show that the analytic function tends to zero at infinity. Then, by virtue of Liouville's theorem, it is identically zero in the entire plane λ. Thus, we obtain $\Omega^-(\lambda)$, $V^+(\lambda)$. With the aid of the Fourier transform, we restore the stress σ_y at the continuation of the cut and the displacement v of its edges, corresponding to the initial boundary value problem

$$\sigma_y = -P\delta(x + l) + \frac{P}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{k_2 - \lambda}}\, [\alpha(\lambda)e^{ik_1 l} + \beta(\lambda)\, e^{i\lambda_R l}]\, e^{i\lambda x}\, d\lambda \tag{1.13}$$

$$v = \frac{iP}{4\pi\mu(1 - k_1^2/k_2^2)}\int_{-\infty}^{\infty} \frac{1}{F(\lambda)(k_2 - \lambda)\sqrt{k_2 + \lambda}}\, [e^{i\lambda l}\sqrt{k_2 - \lambda} - \alpha(\lambda)e^{ik_1 l} - \beta(\lambda)e^{i\lambda_R l}]\, e^{i\lambda x}\, d\lambda \tag{1.14}$$

We determine the stress intensity factor K which presents a fundamental interest in fracture mechanics. Making use of the condition at the end of the crack

$$\sigma_y = K/\sqrt{2\pi x} \quad \text{for} \quad x \to +0$$

we have

$$\Omega^-(\lambda) = \frac{1}{2\pi}\int_0^{\infty}\left(\frac{\sigma_y}{2\mu}\right)_{y=0} e^{-i\lambda x}\, dx = \frac{K}{4\pi\mu\sqrt{2}}\, e^{-i\pi/4}(\lambda)^{-1/2} \tag{1.15}$$

In this connection we consider that λ tends to infinity remaining in the lower half-plane. On the other hand, for $\lambda \to \infty$ we obtain

$$\Omega^-(\lambda) = -\frac{P\sqrt{k_1}}{4\pi\mu}(iMe^{ik_1 l} + Ne^{i\lambda_R l})(\lambda)^{-1/2} \tag{1.16}$$

$$M = \sqrt{\frac{c_1 - c_2}{c_2}}\left[1 - \frac{\sqrt{c_2(c_1 - c_R)} - \sqrt{c_R(c_1 - c_2)}}{\sqrt{c_1(c_2 - c_R)}}\right]$$

$$N = -\sqrt{\frac{c_1 - c_R}{c_R}}$$

Taking into account the previously omitted experimental factor, from formulas (1.15), (1.16) we finally obtain

$$K = Re\,[\overline{K}(\omega)\,e^{-i\omega t}] = -P\,\sqrt{k_1}\,Re\,[(1+i)\,(iMe^{ik_1 l} + Ne^{i\lambda_R l})\,e^{-i\omega t}] \qquad (1.17)$$

2. The nonstationary problem. The stress intensity factor at the vertex of the crack for the nonstationary case is computed with the formula [2]

$$K(t) = \frac{1}{2\pi}\,Re\left[\int_{-\infty}^{\infty} \overline{K}(\omega)\,e^{-i\omega t}\,d\omega\right] \qquad (2.1)$$

Here the function $\overline{K}(\omega)$ is given by the formula (1.17). Substituting (1.17) into (2.1) and evaluating the integral, we obtain

$$K(t) = -\frac{PQ}{(t - l/c_1)^{3/2}} \quad \text{for} \quad t > \frac{l}{c_1}$$
$$K(t) = 0 \quad \text{for} \quad t < \frac{l}{c_1} \qquad (2.2)$$
$$(Q = M/\sqrt{2\pi c_1})$$

Considering the obtained result as a Green's function, we find the stress intensity factor in the case of an arbitrary distribution of the load on the gap

$$K(t) = -Q\int_0^{\infty}\int_{x/c_1}^{t} \frac{p(x,t)}{(t - x/c_1)^{3/2}}\,dt\,dx \qquad (2.3)$$

3. Examples. 1. Assume that the function $p(x, t)$ has the form

$$p(x,t) = -P\delta(x - l)\,H(t) \qquad (3.1)$$

Here $H(t)$ is the Heaviside unit function, $\delta(x)$ is Dirac's delta function and P is some constant. The solution of this self-similar problem, according to (2.3), is

$$K(t) = -\frac{2PQ}{(t - l/c_1)^{1/2}} \qquad (3.2)$$

In particular, for $l \to 0$ we have

$$K(t) = -\frac{2PQ}{\sqrt{t}} \qquad (3.3)$$

This solution coincides with the solution given in [6], if in the latter we set $c \to 0$, $T = -P$, where c is the crack propagation rate.

2. Assume that the load is given in the form

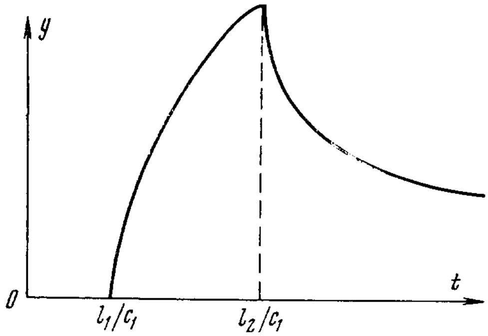

Fig. 2

$$p(x,t)=\begin{cases} -PH(t) & \text{for } l_1<x<l_2 \\ 0 & \text{for } x<l_1,\ x>l_2 \end{cases} \tag{3.4}$$

This is also a self-similar problem and it has the following solution:

$$K(t)=-4c_1QP\,\mathrm{Re}\,[\sqrt{t-l_1/c_1}-\sqrt{t-l_2/c_1}] \tag{3.5}$$

It follows from (3.5) that for $l_1/c_1<t<l_2/c_1$, the stress intensity factor is directly proportional to $(t-l_1/c_1)^{1/2}$, while for $t>l_2/c_1$ it is directly proportional to

$$(\sqrt{t-l_1/c_1}-\sqrt{t-l_2/c_1}).$$

The graph of the function $y=\mathrm{Re}\,(\sqrt{t-l_1/c_1}-\sqrt{t-l_2/c_1})$ is given in Fig. 2.

3. For the function $p(x,t)$ of the form

$$p(x,t)=\begin{cases} -P\delta(t) & \text{for } l_1<x<l_2 \\ 0 & \text{for } x<l_1,\ x>l_3 \end{cases} \tag{3.6}$$

the solution, according to the general formula (2.3), is

$$K(t)=-2c_1PQ\,\mathrm{Re}\left[\frac{\sqrt{t-l_2/c_1}-\sqrt{t-l_1/c_1}}{\sqrt{(t-l_1/c_1)(t-l_2/c_1)}}\right] \tag{3.7}$$

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